

N3 Math Marking Guide

maTH(ə)'matiks
DIFFERENTIAL CALCULUS

N3-Differential Calculus-Answers

1. $y = \frac{1}{x} + 2\sqrt{x}$

$$y = x^{-1} + 2x^{\frac{1}{2}} \quad \checkmark$$

$$\frac{dy}{dx} = -x^{-2} + 2(0,5)x^{-\frac{1}{2}} \quad \checkmark$$

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{\sqrt{x}} \quad \checkmark$$

2. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2x + h - 3)}{h}$$

$$= \lim_{h \rightarrow 0} (2x + h - 3)$$

$$= 2x - 3 \quad \dots$$

3.1. $y = \left(x^2 - \frac{1}{x^2}\right)^2$

$$y = x^4 - 2 + \frac{1}{x^4}$$

$$= x^4 - 2 + x^{-4}$$

3.2. $D_x \left[\frac{x^3 - 1}{x - 1} \right]$

$$= D_x \left[\frac{(x-1)(x^2+x+1)}{x-1} \right]$$

$$= D_x [x^2 + x + 1]$$

$$= 2x + 1$$

3.3. $(t^2 + 1)(t^{-2} - 1) = 1 - t^2 + t^{-2} - 1$

$$\therefore \frac{d}{dt} = -2t - 2t^{-3}$$

$$\therefore \frac{dy}{dx} = 4x^3 - 4x^{-5}$$

3.4. If $y = 2x^3 - 4x^2 + 2x - 1$ then $\frac{dy}{dx} = 6x^2 - 8x + 2$

$$\therefore \frac{d}{dx}(6x^2 - 8x + 2) = 12x - 8$$

3.5. $\frac{d}{d\theta}$ if $h(\theta) = \theta^{\sqrt{2}} + \theta - \theta^{-\sqrt{2}}$ $\therefore \frac{d}{d\theta} = \sqrt{2}\theta^{\sqrt{2}-1} + 1 + \sqrt{2}\theta^{-\sqrt{2}-1}$

4.1. $h'(x) = 3x^2 - 14x + 14$

4.2. At B, $h'(x) = 0 \quad \therefore 3x^2 - 14x + 14 = 0 \Rightarrow x = \frac{14 \pm \sqrt{(-14)^2 - 4(3)(14)}}{2(3)} = \frac{14 \pm \sqrt{28}}{6}$

$$\therefore x = 1,45 \text{ or } x = 3,22 \dots \text{n/a}$$

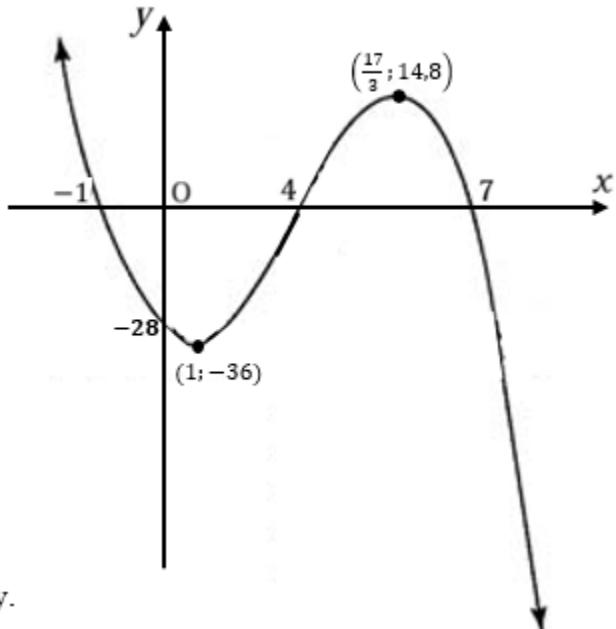
$$\therefore \text{x-coordinate of B: } x = 1,45$$

- 4.3. Since given that $x = 1$ for point A $\Rightarrow x - 1$ is a factor of $h(x)$
and $\frac{h(x)}{x-1} = x^2 - 6x + 8 \therefore h(x) = (x-1)(x-2)(x-4).$
 $\therefore C(4; 0)$...it being the furthest x -intercept to the right.

- 5.1. For the sketch of $f(x) = -x^3 + 10x^2 - 17x - 28$:

- i) y -intercept: -28
- ii) Roots: 1st find one factor using factor thrm
 $(x+1)$ is a factor ...since
 $f(-1) = -(-1)^3 + 10(-1)^2 - 17(-1) - 28 = 0$
Using long division (or synthetic division)
 $\Rightarrow \frac{f(x)}{x+1} = -x^2 + 11x - 28$

- $\therefore f(x) = (x+1)(x-7)(-x+4).$
- $\therefore x$ -intercepts: $x = -1$ or $x = 4$ or $x = 7$
- iii) T. points: Let $f'(x) = 0$
 $\therefore -3x^2 + 20x - 17 = 0$
 $\therefore 3x^2 - 20x + 17 = 0$
 $\therefore (3x-17)(x-1) = 0 \therefore x = 1$ or $x = \frac{17}{3}$
and with $y = -36$ or $y = 14,8$ respectively.

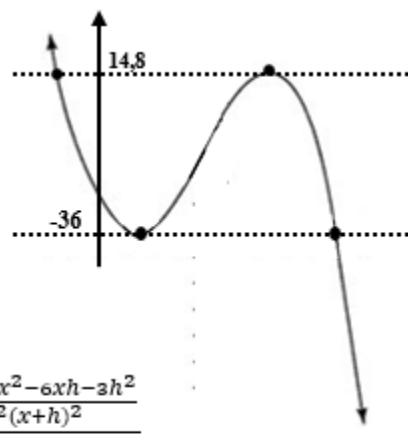


- 5.2. Gradient of $f(x) = f'(x) = -3x^2 + 20x - 17$ and since the equation is quadratic in nature,
the maximum gradient will be at the parabola's turning point, viz. at $x = -\frac{b}{2a} = -\frac{20}{2(-3)}$
i.e. at $x = -3\frac{1}{3}$ (or $x = 3,\dot{3}$) (Note: Largest gradient implies a positive gradient and
from the sketch it can be seen that this can only happen between $x = 1$ or $x = \frac{17}{3}$).

5.3. $-x^3 + 10x^2 - 17x = 25 + k$
 $\Rightarrow -x^3 + 10x^2 - 17x - 28 = k - 3$

- 5.3.1. For two roots only:

OR
$$\begin{cases} k - 3 = 14,8 \therefore k = 17,8 \\ k - 3 = -36 \therefore k = -33 \end{cases}$$



- 5.3.2. For one roots only: $k < -33$ or $k > 17,8$, $k \in R$

(Note: Not asked, but for three roots: $-33 < k < 17,8$)

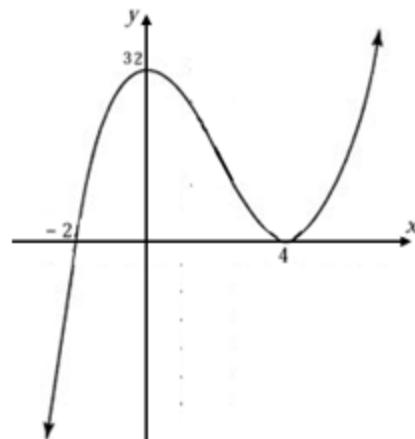
$$6. f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{3x^2 - 3x^2 - 6xh - 3h^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(-6x-3h)}{x^2h(x+h)^2} = \lim_{h \rightarrow 0} \frac{-6x-3h}{x^2(x+h)^2} = \frac{-6x}{x^4} = -\frac{6}{x^3}$$

7.1. $D_x \left[\frac{1}{4\sqrt{x}} - \frac{3x}{\sqrt{x^3}} \right] = D_x \left[\frac{x^{-\frac{1}{2}}}{4} - 3x^{1-\frac{3}{2}} \right] = -\frac{x^{-\frac{3}{2}}}{8} + \frac{3x^{-\frac{5}{2}}}{2} = \frac{11}{8\sqrt{x^3}}$

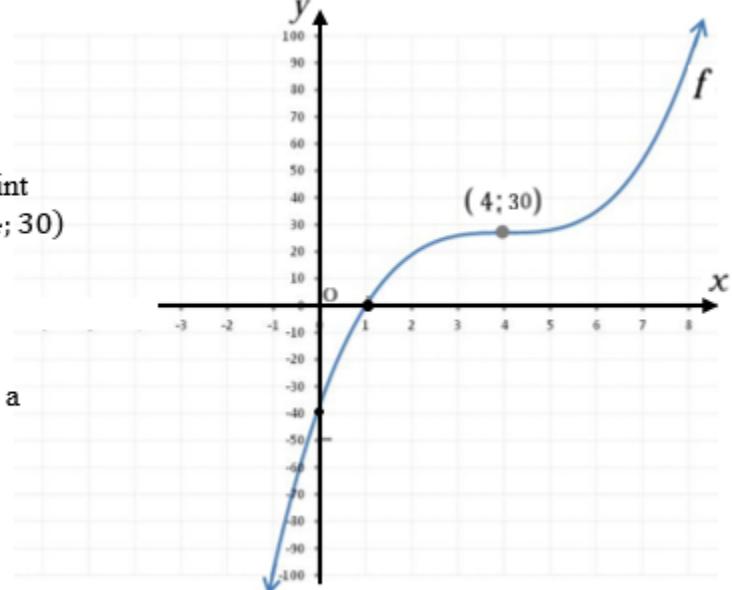
7.2. $y^2 = \frac{x^2 - 2x}{4x - 8} = \frac{x(x-2)}{4(x-2)} = \frac{x}{4} \therefore x = 4y^2 \therefore \frac{dx}{dy} = 8y. \quad (\text{Note: It is } \frac{dx}{dy} \text{ and not } \frac{dy}{dx})$

- 8.
- $g(0) = 32 \Rightarrow d = 32$ (y-intercept)
 - $g(-2) = 0$ and $g(4) = 0 \Rightarrow$ x-intercepts
i.e. at $x = -2$ and $x = 4$
 - $g'(0) = 0$ and $g'(4) = 0 \Rightarrow$ Turning pts
i.e. at $x = 0$ and $x = 4$



9. The graph of f .

- $f(0) = -40 \Rightarrow$ y-intercept
- $f'(4) = 0 \Rightarrow$ a stationary pt.
at $x = 4$ and
- $f(4) = 30 \Rightarrow y = 30$ at this point
- $f''(4) = 0 \Rightarrow$ that this point $(4; 30)$
is also the inflection point.
- $f(x)$ has only one real root
 \Rightarrow graph is neither a linear nor
quadratic function, with $a > 0$ with a
shape as shown in the sketch.



10. (**Note:** When expanding a binomial, and especially one which is raised to the power of 3, as in this case with $2(x + h)^3$, it is most often quicker and less prone to making errors by first expanding the expression rather than expanding it in the steps while using the definition in determining the derivative. This will be done here).

$$\begin{aligned} 2(x + h)^3 &= 2(x + h)(x + h)^2 = (2x + 2h)(x^2 + 2xh + h^2) \\ &= 2x^3 + 4x^2h + 2xh^2 + 2x^2h + 4xh^2 + 2h^3 = 2x^3 + 6x^2h + 6xh^2 + 2h^3 \dots (\text{can also use Pascal's } \Delta \text{ or the binomial theorem to expand the trinomial}) \end{aligned}$$

$$\begin{aligned} \text{Now, } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} = \lim_{h \rightarrow 0} \frac{h(6x^2 + 6xh + 2h^2)}{h} \\ &= \lim_{h \rightarrow 0} 6x^2 + 6xh + 2h^2 = 6x^2 \end{aligned}$$

11. $f(x) = 3x^2 + 2$
 $\therefore f(x+h) = 3(x+h)^2 + 2$
 $= 3x^2 + 6xh + 3h^2 + 2 \checkmark$
 $\therefore \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 2 - 3x^2 - 2}{h} \checkmark$
 $= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h}$
 $= \lim_{h \rightarrow 0} \frac{h(6x + 3h)}{h} \checkmark$
 $= \lim_{h \rightarrow 0} (6x + 3h)$
 $= 6x \checkmark$

12. $y = 2\sqrt{x} + \frac{3}{x} + p^6$
 $= 2x^{\frac{1}{2}} + 3x^{-1} + p^6 \checkmark$
 $\therefore \frac{dy}{dx} = 2 \times \frac{1}{2} x^{-\frac{1}{2}} - 3x^{-2} + 0 \quad \checkmark \quad \checkmark$
 $= \frac{1}{x^{\frac{1}{2}}} - \frac{3}{x^2}$
 $= \frac{1}{\sqrt{x}} - \frac{3}{x^2} \quad \checkmark$

13. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x$
 $\therefore \frac{dy}{dx} = x^2 - x - 2 \checkmark$
At the turning points $\frac{dy}{dx} = 0$
 $\therefore x^2 - x - 2 = 0$
 $(x-2)(x+1) = 0$
 $\therefore x = 2 \text{ or } x = -1$
 $\checkmark \quad \checkmark$

14. $y = x^{\frac{1}{2}} - \frac{3}{x^2}$
 $y = x^{\frac{1}{2}} - 3x^{-2} \checkmark$
 $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + 6x^{-3} \quad \checkmark \quad \checkmark$
 $= \frac{1}{2\sqrt{x}} + \frac{6}{x^3} \quad \checkmark$

$$\begin{aligned}
 15. \quad & 2x^2 + 4x + 7 \\
 & = 2(x^2 + 2x) + 7 \quad \checkmark \\
 & = 2(x^2 + 2x + 1 - 1) + 7 \\
 & = 2(x^2 + 2x + 1) - 2 + 7 \quad \checkmark \\
 & = 2(x + 1)^2 + 5
 \end{aligned}$$

This has a minimum value when the bracket is zero.

The minimum value of the polynomial is 5 when $x = -1$. \checkmark
Or

If $y = 2x^2 + 4x + 7$

then $\frac{dy}{dx} = 4x + 4 \quad \checkmark$

For minimum value: $\frac{dy}{dx} = 0$

$$\therefore 4x + 4 = 0 \quad \checkmark$$

$$\therefore 4x = -4$$

$$\therefore x = -1$$

$$\begin{aligned}
 \text{If } x = -1, \text{ then } y &= 2(-1)^2 + 4(-1) + 7 \\
 &= 2 - 4 + 7 \\
 &= 5
 \end{aligned}$$

\therefore minimum value is 5 \checkmark

Or

$$y = 2x^2 + 4x + 7$$

Turning point: $x = -\frac{b}{2a}$

$$= -\frac{4}{2(2)}$$

$$= -1 \quad \checkmark$$

$$\begin{aligned}
 \text{If } x = -1, \text{ then } y &= 2(-1)^2 + 4(-1) + 7 \\
 &= 2 - 4 + 7 \\
 &= 5
 \end{aligned}$$

\therefore minimum value is 5 \checkmark

$$\begin{aligned}
 16. \quad & y = \frac{x^4}{4} + (\cos x)^0 - 3p + \frac{9}{\sqrt[3]{x^2}} \\
 & y = \frac{1}{4}x^4 + 1 - 3p + 9x^{-\frac{2}{3}} \quad \checkmark \quad \checkmark \\
 & \frac{dy}{dx} = x^3 - 6x^{-\frac{5}{3}} \quad \checkmark \quad \checkmark \\
 & = x^3 - \frac{6}{\sqrt[3]{x^5}} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & \lim_{h \rightarrow 2} \frac{h^3 + h^2 - 6h}{h^2 - 4} \\
 & = \lim_{h \rightarrow 2} \frac{h(h^2 + h - 6)}{(h-2)(h+2)} \quad \checkmark \quad \checkmark \\
 & = \lim_{h \rightarrow 2} \frac{h(h+3)(h-2)}{(h-2)(h+2)} \\
 & = \lim_{h \rightarrow 2} \frac{h(h+3)}{(h+2)} \quad \checkmark \\
 & = \frac{10}{4} = 2,5 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 18. \quad y &= \frac{x^4 - a^2}{x^2 + a} \\
 &= \frac{(x^2 + a)(x^2 - a)}{x^2 + a} \quad \checkmark \\
 &= x^2 - a \quad \checkmark
 \end{aligned}$$

$$\frac{dy}{dx} = 2x \quad \checkmark$$

If students use the quotient rule and didn't simplify, allocate 2 marks

19. B

$$\begin{aligned}
 20.1. \quad f(x) &= \frac{x^2 - 1}{x+1} \\
 \therefore f(x) &= \frac{(x-1)(x+1)}{(x+1)} \quad \checkmark \\
 \therefore f(x) &= x-1 \quad \checkmark \\
 \text{then} \\
 \therefore f'(x) &= 1 \quad \checkmark
 \end{aligned}$$

If $f'(x)$, the derivative sign, is left out - penalise the student. The student will lose 1 mark. Therefore, the student will then reward 3 out of 4 marks.

Alternative answer:

$$\begin{aligned}
 f(x) &= \frac{x^2 - 1}{x+1} \\
 \therefore f'(x) &= \frac{2x(x-1) - (x^2 - 1)}{(x+1)^2} \quad \checkmark \\
 &= \frac{(x+1)[2x - (x-1)]}{(x+1)^2} \\
 &= \frac{x+1}{x+1} \\
 &= 1 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 20.2. \quad f(x) &= 2x^{\frac{1}{3}} - \frac{3}{x} + b(ab)^0 \\
 \therefore f(x) &= 2x^{\frac{1}{3}} - 3x^{-1} + b \quad \checkmark \\
 \therefore f'(x) &= \frac{2}{3}x^{-\frac{2}{3}} + 3x^{-2} \quad \checkmark \quad \checkmark \\
 &= \frac{2}{3x^{\frac{2}{3}}} + \frac{3}{x^2} \\
 &= \frac{2}{3\sqrt[3]{x^2}} - \frac{3}{x^2} \quad \checkmark \quad \checkmark
 \end{aligned}$$

If the B was carried to the final answer - penalise by 1 mark
 The first term in the final answer should be in surd form. The student will lose a half mark for not writing it in surd form.

$$21.1. \quad y = 2\sqrt{x} - \frac{2}{x}$$

$$= 2x^{\frac{1}{2}} - 2x^{-1} \quad \checkmark \quad \checkmark$$

$\checkmark \quad \checkmark$

$$\therefore \frac{dy}{dx} = x^{-\frac{1}{2}} + 2x^{-2}$$

$$= \frac{1}{\sqrt{x}} + \frac{2}{x^2}$$

$\checkmark \quad \checkmark$

- ✓ rewriting square root
- ✓ rewriting a fraction
- ✓ differentiate function
- ✓ differentiate function
- ✓ rewriting function
- ✓ rewriting function

$$21.2. \quad y = \frac{x^8 - 1}{x^4 + 1}$$

$$y = \frac{(x^4 - 1)(x^4 + 1)}{x^4 + 1} \quad \checkmark$$

$$y = x^4 - 1 \quad \checkmark$$

$$\frac{dy}{dx} = 4x^3 \quad \checkmark$$

- ✓ $(2x\sqrt{ })$ simplify fraction
- ✓ differentiation

$$\begin{aligned}
 22. \quad y &= (x^2 - 3)(x + 1) \\
 y &= x^3 + x^2 - 3x - 3 \\
 f'(x) &= 3x^2 + 2x - 3 \\
 f'(2) &= 3(2)^2 + 2(2) - 3 \\
 &= 3(2)^2 + 4 - 3 \\
 &\equiv 13
 \end{aligned}$$

- ✓ multiplication
- ✓ differentiation
- ✓ substitution

$$\begin{aligned}
 23. \quad & \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-4x^2 - 8xh - 4h^2 - (-4x^2)}{h} \quad \checkmark \\
 &= \lim_{h \rightarrow 0} \frac{-4h(2x+h)}{h} \quad \checkmark \quad \checkmark \\
 &= -4(2x+0) \\
 &= -8x \quad \checkmark
 \end{aligned}$$

- ✓ multiplying out $2(x+h)^2$ and substituting
- ✓ factorising
- ✓ simplifying
- ✓ answer

$$\begin{aligned}
 24.1. \quad y &= \frac{1}{x^4} - 4\sqrt[4]{x} \\
 y &= x^{-4} - 4x^{\frac{1}{4}} & \checkmark \\
 \frac{dy}{dx} &= -4x^{-5} - 4 \times \frac{1}{4} x^{\frac{1}{4}-1} & \checkmark \\
 &= -4x^{-5} - x^{-\frac{3}{4}} & \checkmark \\
 &= \frac{-4}{x^5} - \frac{1}{\sqrt[4]{x^3}} & \checkmark
 \end{aligned}$$

- ✓ rewriting by applying exponential rules
- ✓✓ applying differentiation rule
- ✓ writing it as positive exponent and in surd form

24.2 $y = \frac{x(x^2 - 1)}{x-1}$ ✓
✓ factorise
 $y = \frac{x(x+1)(x-1)}{x-1}$
✓ differentiate

$$y = x^2 + x$$

$$\frac{dy}{dx} = 2x + 1 \quad \checkmark \quad \checkmark$$

25.1. $y = -\frac{2}{x} - 2\sqrt{x}$

$$y = -2x^{-1} - 2x^{1/2} \quad \checkmark$$

$$\frac{dy}{dx} = 2x^{-2} - x^{-1/2} \quad \checkmark$$

$$= \frac{2}{x^2} - \frac{1}{\sqrt{x}} \quad \checkmark$$

25.2. $y = \frac{x^5 + x^3}{5x^2}$ ✓
 $y = \frac{1}{5}x^3 + \frac{1}{5}x$
 $\frac{dy}{dx} = \frac{3}{5}x^2 + \frac{1}{5} \quad \checkmark \quad \checkmark$

26. $y = x^3 - 9x$

$$\frac{dy}{dx} = 3x^2 - 9$$

$$\therefore 3(x^2 - 3) = 0 \quad \checkmark$$

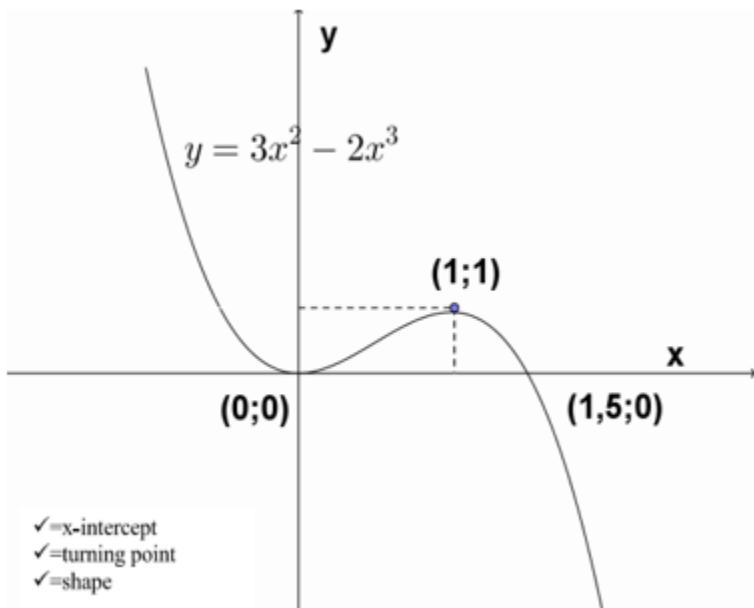
$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

\therefore x-coordinate of P = $-\sqrt{3}$ ✓
x coordinate of Q = $\sqrt{3}$ ✓

27.1. $f(x) = 3x^2 - 2x^3$ ✓
 $f'(x) = 6x - 6x^2$
 $\therefore 6x - 6x^2 = 0$
 $6x(1-x) = 0$
 $\therefore x = 0 \text{ or } x = 1$
 $f(0) = 0 \quad f(1) = 3 - 2 = 1$
 $\therefore (0;0) \quad \text{or} \quad (1;1)$ ✓✓

27.2.



$$28. \quad \therefore \frac{dy}{dx} = \frac{d}{dx} \left(3 - 2x - \frac{4}{x^2} \right) \quad \checkmark$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} 3 - \frac{d}{dx} 2x - \frac{d}{dx} 4x^{-2} \quad \checkmark$$

$$= 0 - 2 - 4(-2x^{-3})$$

$$= -2 + \frac{8}{x^3} \quad \checkmark$$

$$29. \quad \therefore f(x+h) = 10(x+h)^2$$

$$= 10x^2 + 20xh + 10h^2$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{10x^2 + 20xh + 10h^2 - 10x^2}{h} \quad \checkmark$$

$$= \lim_{h \rightarrow 0} \frac{h(20x+10h)}{h}$$

$$= \lim_{h \rightarrow 0} (20x+10h) \quad \checkmark$$

$$= 20x + 0$$

$$= 20x \quad \checkmark$$

$$30.1. \quad y = x^3 + 6x^2 + 9x$$

$$\frac{dy}{dx} = 3x^2 + 12x + 9 \quad \checkmark$$

Let $y=0$

$$\therefore 3x^2 + 12x + 9 = 0$$

$$3(x^2 + 4x + 3) = 0$$

$$(x+1)(x+3) = 0 \quad \checkmark \quad \checkmark$$

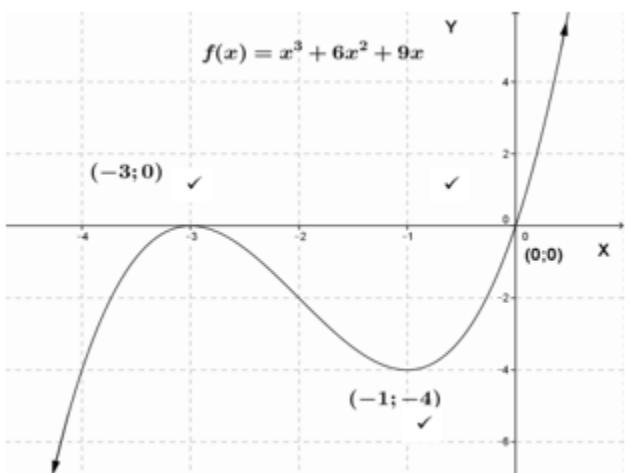
$$x = -1 \text{ or } x = -3$$

$$f(-1) = -(-1)^3 - 6(-1)^2 - 9(-1) = 4$$

$$f(-3) = -(-3)^3 - 6(-3)^2 - 9(-3) = 0 \quad \checkmark \quad \checkmark$$

Turning points $(-3; 0)$ and $(-1; 4)$

30.2.



$$31. \quad y = \frac{1}{x} + 2\sqrt{x}$$

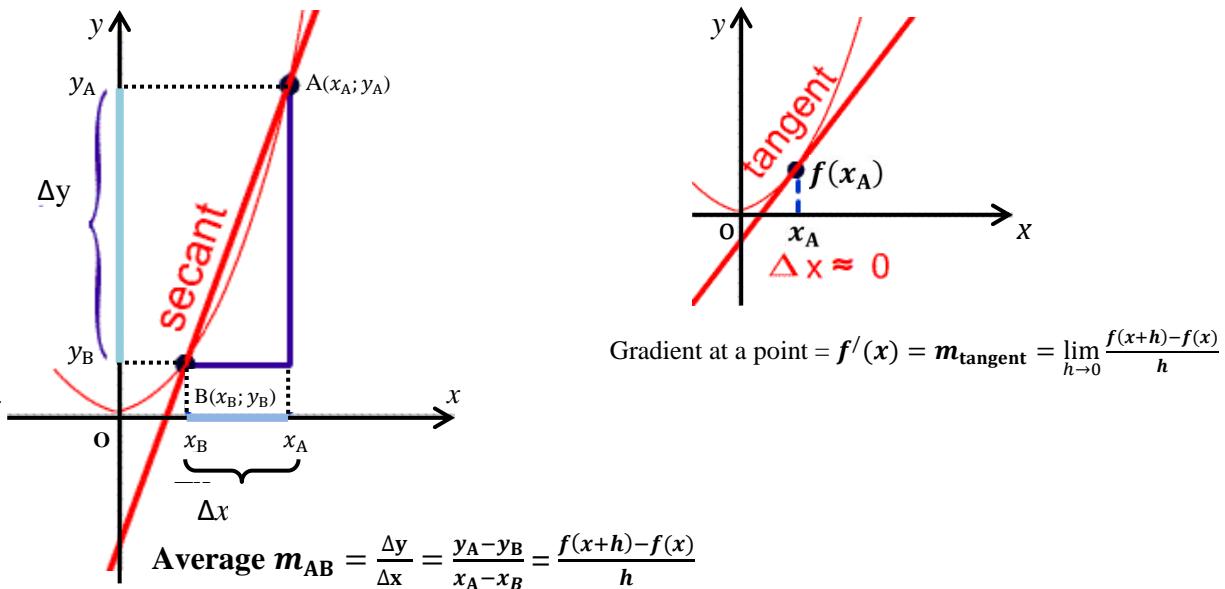
$$y = x^{-1} + 2x^{\frac{1}{2}} \quad \checkmark$$

$$\frac{dy}{dx} = -x^{-2} + 2(0,5)x^{-\frac{1}{2}} \quad \checkmark$$

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{\sqrt{x}} \quad \checkmark \quad \checkmark$$

Part2

Differential Calculus



To excel in this topic, a thorough grasp of the following is essential:

- The average gradient between two points, gradient at a point and the limit-concept.
- The *Power Rule* and the other rules of differentiation.
- Determining the derivatives of functions using first principles, and from applying the rules.
- The cubic function $y = ax^3 + bx^2 + cx + d$, its factors and other properties such as shape, roots (zeros), turning points (critical points) and point of inflexion.
- Practical applications including that of particle motion.

$$\begin{aligned}
 1.1 \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2x + h - 3)}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h - 3) \\
 &= 2x - 3
 \end{aligned}$$

$$\begin{aligned}
 1.2.1 \quad y &= \left(x^2 - \frac{1}{x^2} \right)^2 \\
 &= x^4 - 2 + \frac{1}{x^4} \\
 &= x^4 - 2 + x^{-4} \\
 \therefore \frac{dy}{dx} &= 4x^3 - 4x^{-5}
 \end{aligned}$$

$$\begin{aligned}
 1.2.2 \quad D_x \left[\frac{x^3 - 1}{x - 1} \right] &= D_x \left[\frac{(x-1)(x^2+x+1)}{x-1} \right] \\
 &= D_x [x^2 + x + 1] \\
 &= 2x + 1
 \end{aligned}$$

$$\begin{aligned}
 1.2.3 \quad (t^2 + 1)(t^{-2} - 1) &= 1 - t^2 + t^{-2} - 1 \\
 \therefore \frac{d}{dt} &= -2t - 2t^{-3}
 \end{aligned}$$

1.2.4 If $y = 2x^3 - 4x^2 + 2x - 1$ then $\frac{dy}{dx} = 6x^2 - 8x + 2$

$$\therefore \frac{d}{dx}(6x^2 - 8x + 2) = 12x - 8$$

1.2.5 $\frac{d}{d\theta}$ if $h(\theta) = \theta^{\sqrt{2}} + \theta - \theta^{-\sqrt{2}}$ $\therefore \frac{d}{d\theta} = \sqrt{2}\theta^{\sqrt{2}-1} + 1 + \sqrt{2}\theta^{-\sqrt{2}-1}$

2.1 $h'(x) = 3x^2 - 14x + 14$

2.2 At B, $h'(x) = 0 \therefore 3x^2 - 14x + 14 = 0 \Rightarrow x = \frac{14 \pm \sqrt{(-14)^2 - 4(3)(14)}}{2(3)} = \frac{14 \pm \sqrt{28}}{6}$

$$\therefore x = 1,45 \text{ or } x = 3,22 \dots \text{n/a}$$

$$\therefore x\text{-coordinate of B: } x = 1,45$$

2.3 Since given that $x = 1$ for point A $\Rightarrow x - 1$ is a factor of $h(x)$

$$\text{and } \frac{h(x)}{x-1} = x^2 - 6x + 8 \therefore h(x) = (x-1)(x-2)(x-4).$$

$\therefore C(4; 0)$...it being the furthest x -intercept to the right.

3.1 For the sketch of $f(x) = -x^3 + 10x^2 - 17x - 28$:

i) y-intercept: -28

ii) Roots: 1st find one factor using factor thrm

$(x+1)$ is a factor ...since

$$f(-1) = -(-1)^3 + 10(-1)^2 - 17(-1) - 28 = 0$$

Using long division (or synthetic division)

$$\Rightarrow \frac{f(x)}{x+1} = -x^2 + 11x - 28$$

$$\therefore f(x) = (x+1)(x-7)(-x+4).$$

$\therefore x$ -intercepts: $x = -1$ or $x = 4$ or $x = 7$

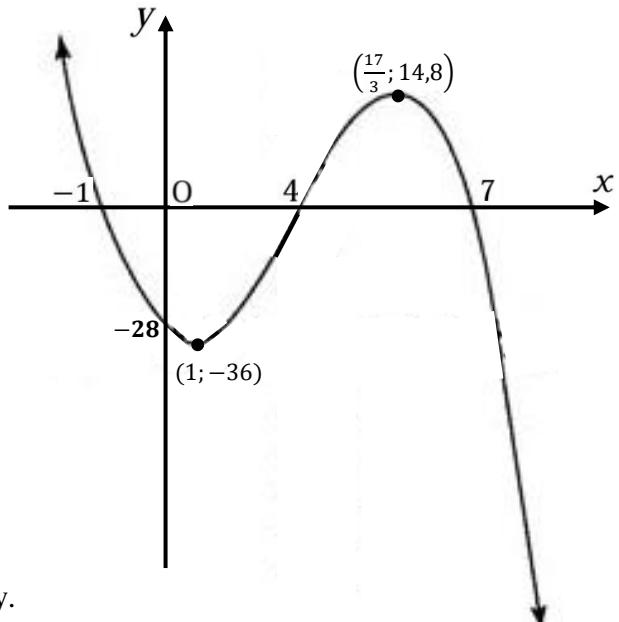
iii) T. points: Let $f'(x) = 0$

$$\therefore -3x^2 + 20x - 17 = 0$$

$$\therefore 3x^2 - 20x + 17 = 0$$

$$\therefore (3x-17)(x-1) = 0 \therefore x = 1 \text{ or } x = \frac{17}{3}$$

and with $y = -36$ or $y = 14,8$ respectively.



3.2 Gradient of $f(x) = f'(x) = -3x^2 + 20x - 17$ and since the equation is quadratic in nature,

$$\text{the maximum gradient will be at the parabola's turning point, viz. at } x = -\frac{b}{2a} = -\frac{20}{2(-3)}$$

i.e. at $x = 3\frac{1}{3}$ (or $x = 3,3$) (**Note:** Largest gradient implies a positive gradient and

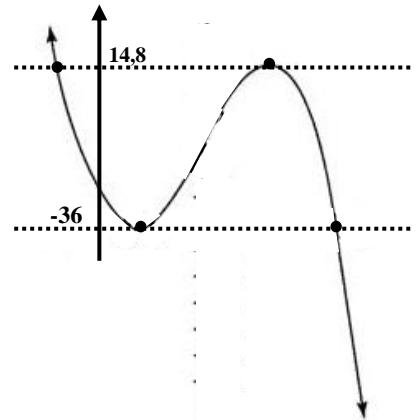
from the sketch it can be seen that this can only happen between $x = 1$ or $x = \frac{17}{3}$).

$$3.3 \quad -x^3 + 10x^2 - 17x = 25 + k \\ \Rightarrow -x^3 + 10x^2 - 17x - 28 = k - 3$$

3.3.1 For two roots only:

OR

$k - 3 = 14,8 \quad \therefore k = 17,8$
$k - 3 = -36 \quad \therefore k = -33$



3.3.2 For one roots only: $k < -33$ or $k > 17,8$, $k \in R$

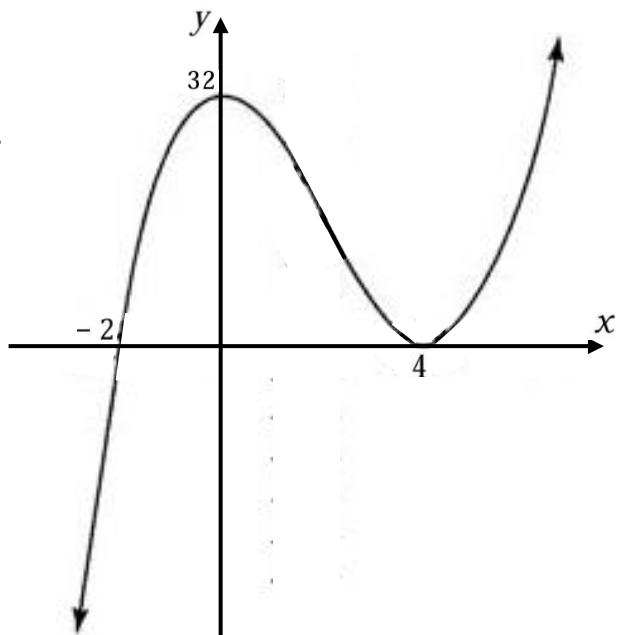
(**Note:** Not asked, but for three roots: $-33 < k < 17,8$)

$$4.1 \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{3x^2 - 3x^2 - 6xh - 3h^2}{x^2(x+h)^2}}{h} \\ = \lim_{h \rightarrow 0} \frac{h(-6x - 3h)}{x^2 h(x+h)^2} = \lim_{h \rightarrow 0} \frac{-6x - 3h}{x^2(x+h)^2} = \frac{-6x}{x^4} = -\frac{6}{x^3}$$

$$4.2.1 \quad D_x \left[\frac{1}{4\sqrt{x}} - \frac{3x}{\sqrt{x^3}} \right] = D_x \left[\frac{x^{-\frac{1}{2}}}{4} - 3x^{1-\frac{3}{2}} \right] = -\frac{x^{-\frac{3}{2}}}{8} + \frac{3x^{-\frac{3}{2}}}{2} = \frac{11}{8\sqrt{x^3}}$$

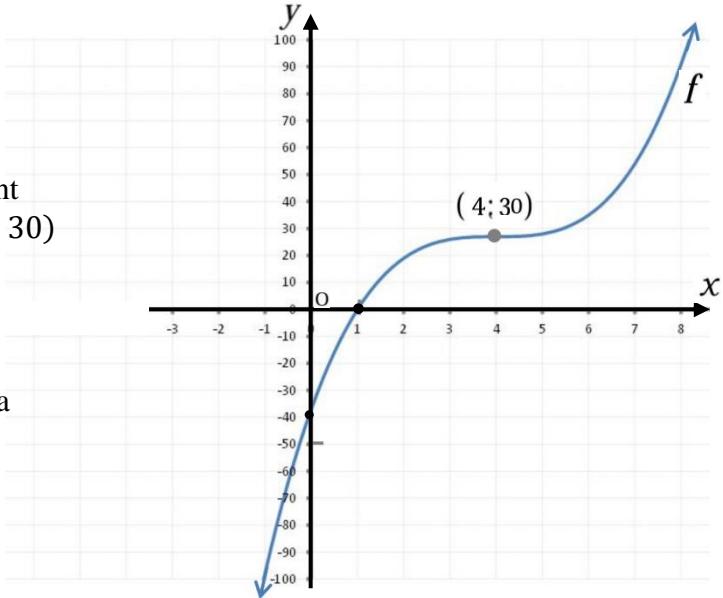
$$4.2.2 \quad y^2 = \frac{x^2 - 2x}{4x - 8} = \frac{x(x-2)}{4(x-2)} = \frac{x}{4} \quad \therefore x = 4y^2 \quad \therefore \frac{dx}{dy} = 8y. \quad (\text{Note: It is } \frac{dx}{dy} \text{ and not } \frac{dy}{dx}).$$

5. i) $g(0) = 32 \Rightarrow d = 32$ (y-intercept)
ii) $g(-2) = 0$ and $g(4) = 0 \Rightarrow$ x-intercepts
i.e. at $x = -2$ and $x = 4$
iii) $g'(0) = 0$ and $g'(4) = 0 \Rightarrow$ Turning pts
i.e. at $x = 0$ and $x = 4$



6. The graph of f .

- i) $f(0) = -40 \Rightarrow$ y-intercept
- ii) $f'(4) = 0 \Rightarrow$ a stationary pt.
at $x = 4$ and
- iii) $f(4) = 30 \Rightarrow y = 30$ at this point
- iv) $f''(4) = 0 \Rightarrow$ that this point $(4; 30)$
is also the inflexion point.
- v) $f(x)$ has only one real root
 \Rightarrow graph is neither a linear nor
quadratic function, with $a > 0$ with a
shape as shown in the sketch.



7.1 (**Note:** When expanding a binomial, and especially one which is raised to the power of 3, as in this case with $2(x + h)^3$, it is most often quicker and less prone to making errors by first expanding the expression rather than expanding it in the steps while using the definition in determining the derivative. This will be done here).

$$\begin{aligned} 2(x + h)^3 &= 2(x + h)(x + h)^2 = (2x + 2h)(x^2 + 2xh + h^2) \\ &= 2x^3 + 4x^2h + 2xh^2 + 2x^2h + 4xh^2 + 2h^3 = 2x^3 + 6x^2h + 6xh^2 + 2h^3 \dots (\text{can also use Pascal's } \Delta \text{ or the binomial theorem to expand the trinomial}) \end{aligned}$$

$$\begin{aligned} \text{Now, } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} = \lim_{h \rightarrow 0} \frac{h(6x^2 + 6xh + 2h^2)}{h} \\ &= \lim_{h \rightarrow 0} 6x^2 + 6xh + 2h^2 = 6x^2 \end{aligned}$$

$$7.2.1 \quad \lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t - 1} = \lim_{t \rightarrow 1} = \lim_{t \rightarrow 1} \frac{(t+2)(t-1)}{t-1} = \lim_{t \rightarrow 1} (t+2) = 3$$

$$7.2.2 \quad \lim_{x \rightarrow \infty} 3^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{3}\right)^x = 0$$

8. i) Roots: (Use factor theorem to find the 1st linear factor of

$$f(x) = x^3 - x^2 - x - 2$$

$$f(1) \neq 0; f(-1) \neq 0$$

$$f(2) = (2)^3 - (2)^2 - (2) - 2 = 0$$

$\therefore (x - 2)$ is a factor. Applying long division, synthetic division or by inspection:

$$\therefore f(x) = (x - 2)(x^2 + x + 1) = 0$$

$$\therefore x = 2 \text{ or } x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} \\ = \frac{-1 \pm \sqrt{-3}}{2} \dots \text{(non real)}$$

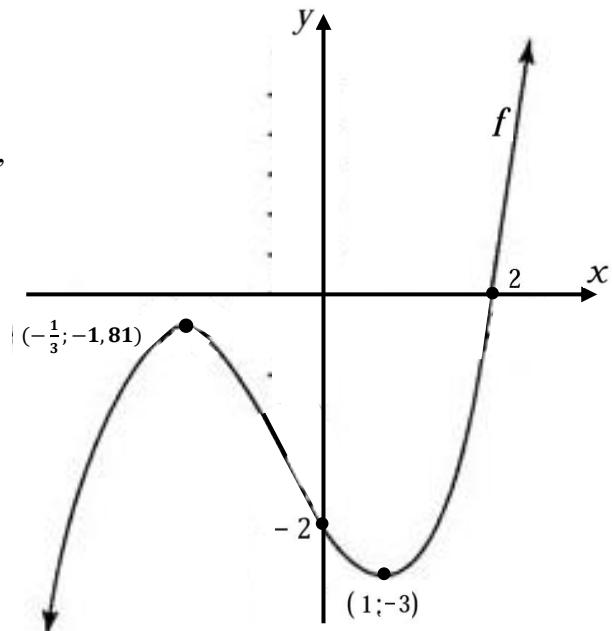
- ii) Stationary points:

$$\text{Set } f'(x) = 0 \therefore 3x^2 - 2x - 1 = 0$$

$$\therefore (3x + 1)(x - 1) = 0$$

$$\therefore x = -\frac{1}{3} \text{ or } x = 1$$

$$\therefore \text{t. pts. } \left(-\frac{1}{3}; -1,81\right) \text{ or } (1; -3)$$



9. (Note): At first glance, it seems impossible to solve, since we have only one x -coordinate ($x = 3$), which lies on both the functions f and g , three unknowns, k , t and p , and only two equations. Often overlooked though, is that we are given the gradient (albeit indirectly) of a tangent, g ($m = 4$) at the point ($x = 3$) and a cubic function, thereby pointing to differentiation).

$$g(x) = x^3 - 2x^2 + kx + 12 \Rightarrow g'(x) = 3x^2 - 4x + k. \therefore 4 = 3(3)^2 - 4(3) + k \\ \therefore k = -11.$$

$$\text{Substituting } (3; p) \text{ into } g: p = (3)^3 - 2(3)^2 - 11(3) + 12 \therefore p = 27 - 18 - 33 + 12 \\ \therefore p = -12$$

$$\text{Substituting } (3; -12) \text{ into } f: -12 = 4(3) + t \therefore t = -24.$$

$$\text{For coordinates of A: Set } x^3 - 2x^2 - 11x + 12 = 4x - 24 \therefore x^3 - 2x^2 - 15x + 36 = 0$$

$$\therefore (x - 3)(x^2 + x - 12) = 0 \dots (\text{we know } 3 \text{ is a root}) \therefore (x - 3)(x - 3)(x + 4) = 0$$

$$\therefore x = -4 \quad (x = 3 \text{ we already have}) \text{ and } y = 4(-4) - 24 = -40$$

$$\therefore A(-4; -40)$$

$$\therefore y = 4x + 13,5$$

10. If $f'(x) = 6x^2 - 6x - 36$, then by using the power rule in ‘reverse’ (so to speak,
 $\Rightarrow f(x) = 2x^3 - 3x^2 - 36x + k$. and given $f(0) = 37 \Rightarrow k = 37$.
 $\therefore f(x) = 2x^3 - 3x^2 - 36x + 37$

11.1 $p(x) = x^3 - x^2 - 8x + 12$. $p(2) = (2)^3 - (2)^2 - 8(2) + 12 = 20 - 20 = 0$
 $\therefore (x - 2)$ is a factor.

11.2 Applying long division, synthetic division or even by inspection: $\frac{p(x)}{x-2} = x^2 + x - 6$
 $\therefore p(x) = (x - 2)(x + 3)(x - 2)$

12.1 x -intercepts (roots): Let $f(x) = 2x^3 - 5x^2 + 4$ and use factor theorem to find the 1st linear factor of $f(x)$
 $f(1) \neq 0$; $f(-1) \neq 0$
 $f(2) = 2(2)^3 - 5(2)^2 + 4 = 20 - 20 = 0 \therefore (x - 2)$ is a factor. Applying long division, synthetic division or even by inspection: $f(x) = (x - 2)(2x^2 - x - 2)$

$$\begin{aligned}\therefore x = 2 \text{ or } x &= \frac{1 \pm \sqrt{(1)^2 - 4(2)(-2)}}{2(2)} \\ &= \frac{1 \pm \sqrt{17}}{4} \quad \therefore x = 2 \text{ or } x = 1,28 \text{ or } x = -0,78\end{aligned}$$

12.2 $f'(x) = 6x^2 - 10x$ and $f''(x) = 12x - 10$. (i) For turning points, let $f'(x) = 0$.
 $\therefore f'(x) = 6x^2 - 10x = 0 \Rightarrow 2x(3x - 5) = 0 \therefore x = \frac{5}{3}$ or $x = 0$
(ii) For point of inflection, let $f''(x) = 0 \therefore 12x - 10 = 0 \therefore x = \frac{5}{6}$

12.3 Gradient (m) = $f'(x) = 6x^2 - 10x \therefore$ gradient at $x = 2 \Rightarrow f'(2) = 6(2)^2 - 10(2) = 4$
Let angle of inclination be $\theta \Rightarrow \tan \theta = m = 4 \therefore \theta = \tan^{-1}(4,0) = 75,96^\circ$.

13.1 $P = 10v - \frac{v^2}{10}$ For maximum profit, let $P' = 0 \Rightarrow 10 - \frac{v}{5} = 0 \therefore v = 50$ km/h.

13.2 Maximum profit (P) = $10v - \frac{v^2}{10} = 10(50) - \frac{50^2}{10} = 250$ cents = R2,50.

14.1 Velocity = $S'(t) = 112 - 32t \dots$ (or $\frac{ds}{dt}$) $\therefore S'(0) = 112 - 32(0) = 112$ m/s

14.2 $\therefore S'(3) = 112 - 32(03) = 16$ m/s

14.3 Maximum height \Rightarrow Let $S'(t) = 0 \Rightarrow 112 - 32t = 0 \therefore t = 3,5$ s
 $S = 112t - 16t^2 \therefore S(3,5) = 112(3,5) - 16(3,5)^2 = 196$ m

15.1 $a = 3t + 2 \Rightarrow$ Velocity: $V(t) = \frac{3}{2}t^2 + 2t + c$ and given $V(0) = 5 \Rightarrow c = 5$

$$\therefore V(t) = \frac{3}{2}t^2 + 2t + 5 \therefore V(2) = \frac{3}{2}(2)^2 + 2(2) + 5 = 15$$

15.2 Since $V(t) = S'(t) \Rightarrow S(t) = \frac{1}{2}t^3 + t^2 + 5t \therefore$ at $t = 2$, $S(2) = \frac{1}{2}(2)^3 + (2)^2 + 5(2) = 18$

16.1 At birth, $t = 0 \therefore$ average mass $B(0) = 0 - 0 + 4,5 = 4,5$ kg.

16.2 For minimum mass, let $B'(t) = 0 \therefore B'(t) = \frac{3t^2}{4500} - \frac{2t}{4500} = 0 \therefore 3t^2 - 20t = 0$

$\therefore t(3t - 20) = 0 \therefore t = 0 \dots \text{n/a}$ or $t = 6\frac{2}{3}$. \therefore Reaches minimum mass after $6\frac{2}{3}$ days

16.3 Set mass equation = birth mass. $B(t) = \frac{t^3}{45000} - \frac{t^2}{4500} + 4,5 = 4,5$ and solve for t .

$$\therefore B(t) = \frac{t^3}{45000} - \frac{t^2}{4500} = 0 \therefore t^3 - 10t^2 = 0 \Rightarrow t^2 - 10 = 0$$

$\therefore t = 0 \text{ n/a } \dots \text{(mass at birth)}$ or $t = 10 \therefore$ Reaches birth mass again after 10 days.

17.1 At present (meaning now) $\Rightarrow t = 0 \therefore n(0) = 20(0)^2 - 200(0) + 500 = 500$ birds.

17.2 Extinct $\Rightarrow n = 0 \therefore n(t) = 20t^2 - 200t + 500 = 0 \therefore t^2 - 10t + 25 = 0$

$\therefore (t - 5)(t - 5) = 0 \therefore t = 5$ or $t = 5 \therefore$ In 5 years time.

17.3 Population to double $\Rightarrow 1000$ birds. $\therefore n(t) = 20t^2 - 200t + 500 = 1000$

$$\therefore t^2 - 10t - 25 = 0 \therefore t = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(-25)}}{2(1)} = \frac{10 \pm \sqrt{200}}{2}$$

$\therefore t = 12,0710 \dots$ or $t = -2,071 \dots \text{n/a}$

\therefore Population should double in ≈ 12 years.

18.1 Velocity $V(t) = S'(t) = 180 - 9t$ and initial $\Rightarrow t = 0 \therefore V(0) = 180 - 9(0) = 180$
i.e. 180 m/s

18.2 a) For maximum, let $S'(t) = 0 \therefore 180 - 9t = 0 \therefore t = 20$. It takes 20 seconds

b) Maximum height = $S(20) = 200 + 180(20) - 4,5(20)^2 = 3800 - 1800 = 2000$ m.

18.3 a) (Note): We'll show two methods here, the 2nd has a slight flaw in its approach as it doesn't consider the missile's change of velocity over time, but gives an excellent approximate.

i) 1st method: Let height for missile's downward displacement = 0 (i.e. missile touches the water) and solve for t . $S(t) = 200 + 180t - 4,5t^2 = 0 \therefore t^2 - 40t - 44,4 = 0$

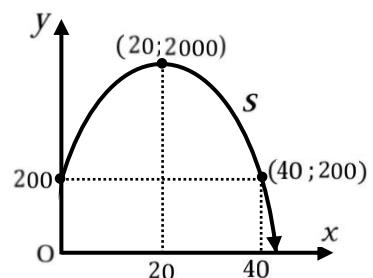
$$\therefore t = \frac{40 \pm \sqrt{(-40)^2 - 4(1)(-44,4)}}{2(1)} = \frac{40 \pm \sqrt{1777,7}}{2} \therefore t = 41,0818 \dots \text{ or } -1,0818 \dots \text{n/a}$$

$\therefore t \approx 41 \therefore$ In air for 41 seconds.

- ii) 2nd method: Going up it took 20 seconds and going down (to the cliff's horizontal level), another 20 s. $\therefore 40$ s + the 200 m still to go. Proportionally, that's \equiv (equivalent) to 1 s.
 $\therefore 41$ seconds.

- b) Velocity: $V(41) = 180 - 9(41) = -189$ m/s(negative as going in opposite direction).

18.4 Graph of $S(t)$ shown on the right.



- 19.1 Given: $y = 8x \cdot \sin 30^\circ - x^2 \cdot \tan 45^\circ = 8\left(\frac{1}{2}\right)x - x^2(1) = 4x - x^2 = y \therefore y' = 4 - 2x$
 For maximum height, let $y' = 0 \Rightarrow 4 - 2x = 0 \therefore$ at $x = 2$ and max. height (y) reached
 $\Rightarrow 4x - x^2 = 4(2) - (2)^2 = 8 - 4 = 4$ km.

- 19.2 Striking the earth $\Rightarrow y = 0 \therefore 4x - x^2 = 0 \therefore x(4 - x) = 0 \therefore x = 4$ or $x \neq 0$
 \therefore Horizontal displacement is 4 km.

- 19.3 Angle of launch = gradient of tangent to function at origin, i.e. where $x = 0 \Rightarrow y' = 0$
 $\therefore m = y' = 4 - 2(0) = 4 \therefore \tan \theta = 4 \Rightarrow \theta = \tan^{-1}(4,0) \therefore \theta = 75,96^\circ$.

20.1 At 11h00 $\Rightarrow t = 2 \therefore D(2) = 80 + \frac{1}{2}(2)^2 - \frac{1}{4}(2)^3 = 80$ m

20.2 Rate $\Rightarrow D'(t)$ at $t = 3$ where $D'(t) = t - \frac{3}{4}t^2 \therefore D'(3) = 3 - \frac{3}{4}(3)^2 = -3,75$ m/h.

- 20.3 Time when inflow of water = outflow \Rightarrow rate of flow = 0 $\Rightarrow D'(t) = 0 \therefore t - \frac{3}{4}t^2 = 0$
 (x by 4): $\therefore 4t - 3t^2 = 0 \therefore t(4 - 3t) = 0 \therefore t = 0 \dots \text{n/a}$ or $t = \frac{4}{3} = 1,3$ hours
 \therefore Time of day: 9h00 + 1,3 hrs i.e. at 10h20.